7.6 Moments, Centers of Mass, and Centroids

- Understand the definition of mass.
- Find the center of mass in a one-dimensional system.
- Find the center of mass in a two-dimensional system.
- Find the center of mass of a planar lamina.
- Use the Theorem of Pappus to find the volume of a solid of revolution.

Mass

In this section, you will study several important applications of integration that are related to mass. Mass is a measure of a body's resistance to changes in motion, and is independent of the particular gravitational system in which the body is located. However, because so many applications involving mass occur on Earth's surface, an object's mass is sometimes equated with its weight. This is not technically correct. Weight is a type of force and as such is dependent on gravity. Force and mass are related by the equation

Force = (mass)(acceleration).

The table below lists some commonly used measures of mass and force, together with their conversion factors.

System of Measurement	Measure of Mass	Measure of Force				
U.S.	Slug	Pound = $(slug)(ft/sec^2)$				
International	Kilogram	Newton = $(kilogram)(m/sec^2)$				
C-G-S	Gram	$Dyne = (gram)(cm/sec^2)$				
Conversions:						
1 pound = 4.44	48 newtons	1 slug = 14.59 kilograms				
1 newton = 0.2	2248 pound	1 kilogram = 0.06852 slug				
1 dyne = 0.000	0002248 pou	nd 1 gram = 0.00006852 slug				
1 dyne = 0.000	001 newton	1 foot = 0.3048 meter				

EXAMPLE 1

Mass on the Surface of Earth

Find the mass (in slugs) of an object whose weight at sea level is 1 pound.

Solution Use 32 feet per second per second as the acceleration due to gravity.

 $Mass = \frac{force}{acceleration}$ $= \frac{1 \text{ pound}}{32 \text{ feet per second per second}}$ $= 0.03125 \frac{\text{pound}}{\text{foot per second per second}}$ = 0.03125 slug

Because many applications involving mass occur on Earth's surface, this amount of mass is called a **pound mass**.

Center of Mass in a One-Dimensional System

You will now consider two types of moments of a mass—the **moment about a point** and the **moment about a line.** To define these two moments, consider an idealized situation in which a mass m is concentrated at a point. If x is the distance between this point mass and another point P, then the **moment of** m about the point P is

Moment = mx



The concept of moment can be demonstrated simply by a seesaw, as shown in Figure 7.53. A child of mass 20 kilograms sits 2 meters to the left of fulcrum P, and an older child of mass 30 kilograms sits 2 meters to the right of P. From experience, you know that the seesaw will begin to rotate clockwise, moving the larger child down. This rotation occurs because the moment produced by the child on the left is less than the moment produced by the child on the right.

Left moment = (20)(2) = 40 kilogram-meters Right moment = (30)(2) = 60 kilogram-meters

To balance the seesaw, the two moments must be equal. For example, if the larger child moved to a position $\frac{4}{3}$ meters from the fulcrum, then the seesaw would balance, because each child would produce a moment of 40 kilogram-meters.

To generalize this, you can introduce a coordinate line on which the origin corresponds to the fulcrum, as shown in Figure 7.54. Several point masses are located on the *x*-axis. The measure of the tendency of this system to rotate about the origin is the **moment about the origin**, and it is defined as the sum of the *n* products $m_i x_i$. The moment about the origin is denoted by M_0 and can be written as

$$M_0 = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n.$$

If M_0 is 0, then the system is said to be in **equilibrium**.



If $m_1x_1 + m_2x_2 + \cdots + m_nx_n = 0$, then the system is in equilibrium. Figure 7.54

For a system that is not in equilibrium, the **center of mass** is defined as the point \bar{x} at which the fulcrum could be relocated to attain equilibrium. If the system were translated \bar{x} units, then each coordinate x_i would become

 $(x_i - \overline{x})$

and because the moment of the translated system is 0, you have

$$\sum_{i=1}^{n} m_i (x_i - \bar{x}) = \sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i \bar{x} = 0.$$

Solving for \overline{x} produces

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\text{moment of system about origin}}{\text{total mass of system}} \cdot$$

When $m_1x_1 + m_2x_2 + \cdots + m_nx_n = 0$, the system is in equilibrium.



The seesaw will balance when the left and the right moments are equal. **Figure 7.53**

Moments and Center of Mass: One-Dimensional System

Let the point masses m_1, m_2, \ldots, m_n be located at x_1, x_2, \ldots, x_n .

1. The moment about the origin is

$$M_0 = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n.$$

2. The center of mass is

$$\overline{x} = \frac{M_0}{m}$$

where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

EXAMPLE 2 The Center of Mass of a Linear System

Find the center of mass of the linear system shown in Figure 7.55.



Figure 7.55

Solution The moment about the origin is

$$M_0 = m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4$$

= 10(-5) + 15(0) + 5(4) + 10(7)
= -50 + 0 + 20 + 70
= 40.

Because the total mass of the system is

m = 10 + 15 + 5 + 10 = 40

the center of mass is

$$\bar{x} = \frac{M_0}{m} = \frac{40}{40} = 1.$$

Note that the point masses will be in equilibrium when the fulcrum is located at x = 1.

Rather than define the moment of a mass, you could define the moment of a *force*. In this context, the center of mass is called the **center of gravity.** Consider a system of point masses m_1, m_2, \ldots, m_n that is located at x_1, x_2, \ldots, x_n . Then, because

force = (mass)(acceleration)

the total force of the system is

 $F = m_1 a + m_2 a + \cdots + m_n a = ma.$

The torque (moment) about the origin is

$$T_0 = (m_1 a)x_1 + (m_2 a)x_2 + \cdots + (m_n a)x_n = M_0 a$$

and the center of gravity is

$$\frac{T_0}{F} = \frac{M_0 a}{ma} = \frac{M_0}{m} = \overline{x}$$

So, the center of gravity and the center of mass have the same location.



In a two-dimensional system, there is a moment about the y-axis M_y and a moment about the x-axis M_x . Figure 7.56



Center of Mass in a Two-Dimensional System

You can extend the concept of moment to two dimensions by considering a system of masses located in the *xy*-plane at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, as shown in Figure 7.56. Rather than defining a single moment (with respect to the origin), two moments are defined—one with respect to the *x*-axis and one with respect to the *y*-axis.

Moment and Center of Mass: Two-Dimensional System

Let the point masses m_1, m_2, \ldots, m_n be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The moment about the y-axis is

$$M_{v} = m_{1}x_{1} + m_{2}x_{2} + \dots + m_{n}x_{n}$$

2. The moment about the x-axis is

$$M_x = m_1 y_1 + m_2 y_2 + \dots + m_n y_n.$$

3. The center of mass (\bar{x}, \bar{y}) (or center of gravity) is

$$\overline{x} = \frac{M_y}{m}$$
 and $\overline{y} = \frac{M_x}{m}$

where

$$m = m_1 + m_2 + \ldots + m_n$$

is the total mass of the system.

The moment of a system of masses in the plane can be taken about any horizontal or vertical line. In general, the moment about a line is the sum of the product of the masses and the *directed distances* from the points to the line.

Moment =
$$m_1(y_1 - b) + m_2(y_2 - b) + \cdots + m_n(y_n - b)$$
 Horizontal line $y = b$
Moment = $m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a)$ Vertical line $x = a$

EXAMPLE 3 The Center of Mass of a Two-Dimensional System

Find the center of mass of a system of point masses $m_1 = 6, m_2 = 3, m_3 = 2$, and $m_4 = 9$, located at

$$(3, -2), (0, 0), (-5, 3), \text{ and } (4, 2)$$

as shown in Figure 7.57.

Solution

m = 6 + 3 + 2 + 9 = 20 $M_y = 6(3) + 3(0) + 2(-5) + 9(4) = 44$ $M_x = 6(-2) + 3(0) + 2(3) + 9(2) = 12$ Momentary

Moment about *y*-axis Moment about *x*-axis

So,

$$\overline{x} = \frac{M_y}{m} = \frac{44}{20} = \frac{11}{5}$$

and

$$\overline{y} = \frac{M_x}{m} = \frac{12}{20} = \frac{3}{5}.$$

The center of mass is $\left(\frac{11}{5}, \frac{3}{5}\right)$.

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You can think of the center of mass (\bar{x}, \bar{y}) of a lamina as its balancing point. For a circular lamina, the center of mass is the center of the circle. For a rectangular lamina, the center of mass is the center of the rectangle.

Figure 7.58

Center of Mass of a Planar Lamina

So far in this section, you have assumed the total mass of a system to be distributed at discrete points in a plane or on a line. Now consider a thin, flat plate of material of constant density called a **planar lamina** (see Figure 7.58). **Density** is a measure of mass per unit of volume, such as grams per cubic centimeter. For planar laminas, however, density is considered to be a measure of mass per unit of area. Density is denoted by ρ , the lowercase Greek letter rho.

Consider an irregularly shaped planar lamina of uniform density ρ , bounded by the graphs of y = f(x), y = g(x), and $a \le x \le b$, as shown in Figure 7.59. The mass of this region is

$$m = (\text{density})(\text{area})$$
$$= \rho \int_{a}^{b} [f(x) - g(x)] dx$$
$$= \rho A$$

where *A* is the area of the region. To find the center of mass of this lamina, partition the interval [a, b] into *n* subintervals of equal width Δx . Let x_i be the center of the *i*th subinterval. You can approximate the portion of the lamina lying in the *i*th subinterval by a rectangle whose height is $h = f(x_i) - g(x_i)$. Because the density of the rectangle is ρ , its mass is



Planar lamina of uniform density ρ Figure 7.59

$$m_i = (\text{density})(\text{area}) = \rho \left[f(x_i) - g(x_i) \right] \Delta x.$$

$$| \underbrace{}_{\text{Density Height}} \underbrace{}_{\text{Width}}$$

Now, considering this mass to be located at the center (x_i, y_i) of the rectangle, the directed distance from the x-axis to (x_i, y_i) is $y_i = [f(x_i) + g(x_i)]/2$. So, the moment of m_i about the x-axis is

Moment = (mass)(distance)

$$= m_i y_i$$

= $\rho [f(x_i) - g(x_i)] \Delta x \left[\frac{f(x_i) + g(x_i)}{2} \right]$

Summing the moments and taking the limit as $n \to \infty$ suggest the definitions below.

Moments and Center of Mass of a Planar Lamina

Let *f* and *g* be continuous functions such that $f(x) \ge g(x)$ on [a, b], and consider the planar lamina of uniform density ρ bounded by the graphs of y = f(x), y = g(x), and $a \le x \le b$.

1. The moments about the x- and y-axes are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2} \right] [f(x) - g(x)] dx$$
$$M_y = \rho \int_a^b x [f(x) - g(x)] dx.$$

2. The center of mass (\bar{x}, \bar{y}) is given by $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$, where $m = \rho \int_a^b [f(x) - g(x)] dx$ is the mass of the lamina.

EXAMPLE 4 The Center of Mass of a Planar Lamina

•••• See LarsonCalculus.com for an interactive version of this type of example.

Find the center of mass of the lamina of uniform density ρ bounded by the graph of $f(x) = 4 - x^2$ and the *x*-axis.

Solution Because the center of mass lies on the axis of symmetry, you know that $\bar{x} = 0$. Moreover, the mass of the lamina is

$$m = \rho \int_{-2}^{2} (4 - x^2) dx$$

= $\rho \left[4x - \frac{x^3}{3} \right]_{-2}^{2}$
= $\frac{32\rho}{3}$.

To find the moment about the *x*-axis, place a representative rectangle in the region, as shown in the figure at the right. The distance from the *x*-axis to the center of this rectangle is

$$y_i = \frac{f(x)}{2} = \frac{4 - x^2}{2}$$

Because the mass of the representative rectangle is

$$\rho f(x) \Delta x = \rho (4 - x^2) \Delta x$$

you have

$$M_x = \rho \int_{-2}^{2} \frac{4 - x^2}{2} (4 - x^2) dx$$

= $\frac{\rho}{2} \int_{-2}^{2} (16 - 8x^2 + x^4) dx$
= $\frac{\rho}{2} \Big[16x - \frac{8x^3}{3} + \frac{x^5}{5} \Big]_{-2}^{2}$
= $\frac{256\rho}{15}$

and \overline{y} is

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$$\bar{v} = \frac{M_x}{m} = \frac{256\rho/15}{32\rho/3} = \frac{8}{5}$$

So, the center of mass (the balancing point) of the lamina is $(0, \frac{8}{5})$, as shown in Figure 7.60.

The density ρ in Example 4 is a common factor of both the moments and the mass, and as such divides out of the quotients representing the coordinates of the center of mass. So, the center of mass of a lamina of *uniform* density depends only on the shape of the lamina and not on its density. For this reason, the point

(\bar{x}, \bar{y}) Center of mass or centroid

is sometimes called the center of mass of a *region* in the plane, or the **centroid** of the region. In other words, to find the centroid of a region in the plane, you simply assume that the region has a constant density of $\rho = 1$ and compute the corresponding center of mass.



The center of mass is the balancing point. **Figure 7.60**









Find the centroid of the region bounded by the graphs of $f(x) = 4 - x^2$ and g(x) = x + 2.

Solution The two graphs intersect at the points (-2, 0) and (1, 3), as shown in Figure 7.61. So, the area of the region is

$$A = \int_{-2}^{1} \left[f(x) - g(x) \right] dx = \int_{-2}^{1} \left(2 - x - x^2 \right) dx = \frac{9}{2}.$$

The centroid (\bar{x}, \bar{y}) of the region has the following coordinates.

$$\bar{x} = \frac{1}{A} \int_{-2}^{1} x[(4 - x^2) - (x + 2)] dx$$

$$= \frac{2}{9} \int_{-2}^{1} (-x^3 - x^2 + 2x) dx$$

$$= \frac{2}{9} \left[-\frac{x^4}{4} - \frac{x^3}{3} + x^2 \right]_{-2}^{1}$$

$$= -\frac{1}{2}$$

$$\bar{y} = \frac{1}{A} \int_{-2}^{1} \left[\frac{(4 - x^2) + (x + 2)}{2} \right] [(4 - x^2) - (x + 2)] dx$$

$$= \frac{2}{9} \left(\frac{1}{2} \right) \int_{-2}^{1} (-x^2 + x + 6)(-x^2 - x + 2) dx$$

$$= \frac{1}{9} \int_{-2}^{1} (x^4 - 9x^2 - 4x + 12) dx$$

$$= \frac{1}{9} \left[\frac{x^5}{5} - 3x^3 - 2x^2 + 12x \right]_{-2}^{1}$$

$$= \frac{12}{5}$$

So, the centroid of the region is $(\bar{x}, \bar{y}) = \left(-\frac{1}{2}, \frac{12}{5}\right)$.

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For simple plane regions, you may be able to find the centroids without resorting to integration.

EXAMPLE 6

The Centroid of a Simple Plane Region

Find the centroid of the region shown in Figure 7.62(a).

Solution By superimposing a coordinate system on the region, as shown in Figure 7.62(b), you can locate the centroids of the three rectangles at

$$\left(\frac{1}{2}, \frac{3}{2}\right)$$
, $\left(\frac{5}{2}, \frac{1}{2}\right)$, and (5, 1).

Using these three points, you can find the centroid of the region.

$$A = \text{area of region} = 3 + 3 + 4 = 10$$

$$\bar{x} = \frac{(1/2)(3) + (5/2)(3) + (5)(4)}{10} = \frac{29}{10} = 2.9$$

$$\bar{y} = \frac{(3/2)(3) + (1/2)(3) + (1)(4)}{10} = \frac{10}{10} = 1$$

So, the centroid of the region is (2.9, 1). Notice that (2.9, 1) is not the "average" of $(\frac{1}{2}, \frac{3}{2}), (\frac{5}{2}, \frac{1}{2})$, and (5, 1).







(b) The centroids of the three rectangles **Figure 7.62**

Theorem of Pappus

The final topic in this section is a useful theorem credited to Pappus of Alexandria (ca. 300 A.D.), a Greek mathematician whose eight-volume Mathematical Collection is a record of much of classical Greek mathematics. You are asked to prove this theorem in Section 14.4.



The volume V is $2\pi rA$, where A is the area of region R. Figure 7.63

THEOREM 7.1 The Theorem of Pappus

Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R, as shown in Figure 7.63. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is

 $V = 2\pi rA$

where A is the area of R. (Note that $2\pi r$ is the distance traveled by the centroid as the region is revolved about the line.)

The Theorem of Pappus can be used to find the volume of a torus, as shown in the next example. Recall that a torus is a doughnut-shaped solid formed by revolving a circular region about a line that lies in the same plane as the circle (but does not intersect the circle).

EXAMPLE 7

Finding Volume by the Theorem of Pappus

Find the volume of the torus shown in Figure 7.64(a), which was formed by revolving the circular region bounded by

$$(x-2)^2 + y^2 =$$

1 about the y-axis, as shown in Figure 7.64(b).



Exploration

Use the shell method to show that the volume of the torus in Example 7 is

$$V = \int_{1}^{3} 4\pi x \sqrt{1 - (x - 2)^2} \, dx.$$

Evaluate this integral using a graphing utility. Does your answer agree with the one in Example 7?

Figure 7.64

Solution In Figure 7.67(b), you can see that the centroid of the circular region is (2, 0). So, the distance between the centroid and the axis of revolution is

r = 2.

Because the area of the circular region is $A = \pi$, the volume of the torus is

$$V = 2\pi rA$$

= $2\pi(2)(\pi)$
= $4\pi^2$
 $\approx 39.5.$

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7.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Center of Mass of a Linear System In Exercises 1–4, find the center of mass of the point masses lying on the *x*-axis.

1.
$$m_1 = 7, m_2 = 3, m_3 = 5$$

 $x_1 = -5, x_2 = 0, x_3 = 3$
2. $m_1 = 7, m_2 = 4, m_3 = 3, m_4 = 8$
 $x_1 = -3, x_2 = -2, x_3 = 5, x_4 = 4$
3. $m_1 = 1, m_2 = 3, m_3 = 2, m_4 = 9, m_5 = 5$
 $x_1 = 6, x_2 = 10, x_3 = 3, x_4 = 2, x_5 = 4$

4.
$$m_1 = 8, m_2 = 5, m_3 = 5, m_4 = 12, m_5 = 2$$

 $x_1 = -2, x_2 = 6, x_2 = 0, x_4 = 3, x_5 = -5$

- 5. Graphical Reasoning
 - (a) Translate each point mass in Exercise 3 to the right four units and determine the resulting center of mass.
 - (b) Translate each point mass in Exercise 4 to the left two units and determine the resulting center of mass.
- **6.** Conjecture Use the result of Exercise 5 to make a conjecture about the change in the center of mass that results when each point mass is translated k units horizontally.

Statics Problems In Exercises 7 and 8, consider a beam of length L with a fulcrum x feet from one end (see figure). There are objects with weights W_1 and W_2 placed on opposite ends of the beam. Find x such that the system is in equilibrium.



- **7.** Two children weighing 48 pounds and 72 pounds are going to play on a seesaw that is 10 feet long.
- **8.** In order to move a 600-pound rock, a person weighing 200 pounds wants to balance it on a beam that is 5 feet long.

Center of Mass of a Two-Dimensional System In Exercises 9–12, find the center of mass of the given system of point masses.

9. $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	-					
$(x_i, y_i) (2, 2) (-3, 1) (1, -4)$ 10. $m_i 10 2 5$ $(x_i, y_i) (1, -1) (5, 5) (-4, 0)$ 11. $m_i 12 6 4.5 15$ $(x_i, y_i) (2, 3) (-1, 5) (6, 8) (2, -2)$	9.	m _i	5	1	3	
m_i 10 2 5 (x_i, y_i) $(1, -1)$ $(5, 5)$ $(-4, 0)$ 11. m_i 12 6 4.5 15 (x_i, y_i) $(2, 3)$ $(-1, 5)$ $(6, 8)$ $(2, -2)$		(x_i, y_i)	(2, 2)	(-3, 1)	(1, -4	.)
m_i 10 2 5 (x_i, y_i) $(1, -1)$ $(5, 5)$ $(-4, 0)$ 11. m_i 12 6 4.5 15 (x_i, y_i) $(2, 3)$ $(-1, 5)$ $(6, 8)$ $(2, -2)$	10					
(x_i, y_i) $(1, -1)$ $(5, 5)$ $(-4, 0)$ 11. m_i 12 6 4.5 15 (x_i, y_i) $(2, 3)$ $(-1, 5)$ $(6, 8)$ $(2, -2)$	10.	m_i	10	2	5	
11. m_i 12 6 4.5 15 (x_i, y_i) (2, 3) (-1, 5) (6, 8) (2, -2)		(x_i, y_i)	(1, -1)	(5, 5)	(-4,0))
m _i 12 6 4.5 15 (x_i, y_i) $(2, 3)$ $(-1, 5)$ $(6, 8)$ $(2, -2)$						
(x_i, y_i) (2, 3) (-1, 5) (6, 8) (2, -2)	11.	m_i	12	6	4.5	15
		(x_i, y_i)	(2, 3)	(-1, 5)	(6, 8)	(2, -2)

12.	m _i	3	4	2	1	6
	(x_i, y_i)	(-2, -3)	(5, 5)	(7, 1)	(0, 0)	(-3, 0)

Center of Mass of a Planar Lamina In Exercises 13–26, find M_x, M_y , and (\bar{x}, \bar{y}) for the laminas of uniform density ρ bounded by the graphs of the equations.

13. $y = \frac{1}{2}x, y = 0, x = 2$ **14.** y = 6 - x, y = 0, x = 0 **15.** $y = \sqrt{x}, y = 0, x = 4$ **16.** $y = \frac{1}{2}x^2, y = 0, x = 2$ **17.** $y = x^2, y = x^3$ **18.** $y = \sqrt{x}, y = \frac{1}{2}x$ **19.** $y = -x^2 + 4x + 2, y = x + 2$ **20.** $y = \sqrt{x} + 1, y = \frac{1}{3}x + 1$ **21.** $y = x^{2/3}, y = 0, x = 8$ **22.** $y = x^{2/3}, y = 4$ **23.** $x = 4 - y^2, x = 0$ **24.** $x = 3y - y^2, x = 0$ **25.** $x = -y, x = 2y - y^2$ **26.** $x = y + 2, x = y^2$

Approximating a Centroid In Exercises 27–30, use a graphing utility to graph the region bounded by the graphs of the equations. Use the integration capabilities of the graphing utility to approximate the centroid of the region.

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27.
$$y = 10x\sqrt{125 - x^3}, y = 0$$

28. $y = xe^{-x/2}, y = 0, x = 0, x = 0$

29. Prefabricated End Section of a Building

$$y = 5\sqrt[3]{400 - x^2}, y = 0$$

30. Witch of Agnesi

$$y = \frac{8}{x^2 + 4}, y = 0, x = -2, x = 2$$

Finding the Center of Mass In Exercises 31–34, introduce an appropriate coordinate system and find the coordinates of the center of mass of the planar lamina. (The answer depends on the position of the coordinate system.)



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- **35. Finding the Center of Mass** Find the center of mass of the lamina in Exercise 31 when the circular portion of the lamina has twice the density of the square portion of the lamina.
- **36. Finding the Center of Mass** Find the center of mass of the lamina in Exercise 31 when the square portion of the lamina has twice the density of the circular portion of the lamina.

Finding Volume by the Theorem of Pappus In Exercises 37–40, use the Theorem of Pappus to find the volume of the solid of revolution.

37. The torus formed by revolving the circle

$$(x-5)^2 + y^2 = 16$$

about the y-axis

38. The torus formed by revolving the circle

$$x^2 + (y - 3)^2 = 4$$

about the x-axis

- **39.** The solid formed by revolving the region bounded by the graphs of y = x, y = 4, and x = 0 about the *x*-axis
- **40.** The solid formed by revolving the region bounded by the graphs of $y = 2\sqrt{x-2}$, y = 0, and x = 6 about the y-axis

WRITING ABOUT CONCEPTS

- **41. Center of Mass** Let the point masses m_1 , m_2, \ldots, m_n be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Define the center of mass (\bar{x}, \bar{y}) .
- **42. Planar Lamina** What is a planar lamina? Describe what is meant by the center of mass (\bar{x}, \bar{y}) of a planar lamina.
- **43. Theorem of Pappus** State the Theorem of Pappus.

HOW DO YOU SEE IT? The centroid of the plane region bounded by the graphs of y = f(x), y = 0, x = 0, and x = 3 is (1.2, 1.4). Is it possible to find the centroid of each of the regions bounded by the graphs of the following sets of equations? If so, identify the centroid and explain your answer.



(d) y = f(x), y = 0, x = 2, and x = 4

Centroid of a Common Region In Exercises 45–50, find and/or verify the centroid of the common region used in engineering.

45. Triangle Show that the centroid of the triangle with vertices (-a, 0), (a, 0), and (b, c) is the point of intersection of the medians (see figure).



- **46. Parallelogram** Show that the centroid of the parallelogram with vertices (0, 0), (a, 0), (b, c), and (a + b, c) is the point of intersection of the diagonals (see figure).
- **47. Trapezoid** Find the centroid of the trapezoid with vertices (0, 0), (0, a), (c, b), and (c, 0). Show that it is the intersection of the line connecting the midpoints of the parallel sides and the line connecting the extended parallel sides, as shown in the figure.



Figure for 47

Figure for 48

- **48. Semicircle** Find the centroid of the region bounded by the graphs of $y = \sqrt{r^2 x^2}$ and y = 0 (see figure).
- **49. Semiellipse** Find the centroid of the region bounded by the graphs of $y = \frac{b}{a}\sqrt{a^2 x^2}$ and y = 0 (see figure).



50. Parabolic Spandrel Find the centroid of the **parabolic spandrel** shown in the figure.

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- **51. Graphical Reasoning** Consider the region bounded by the graphs of $y = x^2$ and y = b, where b > 0.
 - (a) Sketch a graph of the region.
 - (b) Use the graph in part (a) to determine \bar{x} . Explain.
 - (c) Set up the integral for finding M_y . Because of the form of the integrand, the value of the integral can be obtained without integrating. What is the form of the integrand? What is the value of the integral? Compare with the result in part (b).
 - (d) Use the graph in part (a) to determine whether $\overline{y} > \frac{b}{2}$ or $\overline{y} < \frac{b}{2}$. Explain.
 - (e) Use integration to verify your answer in part (d).
- **52. Graphical and Numerical Reasoning** Consider the region bounded by the graphs of $y = x^{2n}$ and y = b, where b > 0 and *n* is a positive integer.
 - (a) Sketch a graph of the region.
 - (b) Set up the integral for finding M_y. Because of the form of the integrand, the value of the integral can be obtained without integrating. What is the form of the integrand? What is the value of the integral and what is the value of x̄?
 - (c) Use the graph in part (a) to determine whether $\overline{y} > \frac{b}{2}$ or

 $\overline{y} < \frac{b}{2}$. Explain.

- (d) Use integration to find \overline{y} as a function of *n*.
- (e) Use the result of part (d) to complete the table.



(f) Find $\lim \overline{y}$.

(g) Give a geometric explanation of the result in part (f).

53. Modeling Data The manufacturer of glass for a window in a conversion van needs to approximate its center of mass. A coordinate system is superimposed on a prototype of the glass (see figure). The measurements (in centimeters) for the right half of the symmetric piece of glass are listed in the table.



- (a) Use Simpson's Rule to approximate the center of mass of the glass.
- (b) Use the regression capabilities of a graphing utility to find a fourth-degree polynomial model for the data.
- (c) Use the integration capabilities of a graphing utility and the model to approximate the center of mass of the glass. Compare with the result in part (a).

54. Modeling Data The manufacturer of a boat needs to approximate the center of mass of a section of the hull. A coordinate system is superimposed on a prototype (see figure). The measurements (in feet) for the right half of the symmetric prototype are listed in the table.



(a) Use Simpson's Rule to approximate the center of mass of the hull section.

0.43

0.33

0

0.48

d

0.50

- (b) Use the regression capabilities of a graphing utility to find fourth-degree polynomial models for both curves shown in the figure. Plot the data and graph the models.
- (c) Use the integration capabilities of a graphing utility and the models to approximate the center of mass of the hull section. Compare with the result in part (a).

Second Theorem of Pappus In Exercises 55 and 56, use the *Second Theorem of Pappus*, which is stated as follows. If a segment of a plane curve C is revolved about an axis that does not intersect the curve (except possibly at its endpoints), the area S of the resulting surface of revolution is equal to the product of the length of C times the distance d traveled by the centroid of C.

- **55.** A sphere is formed by revolving the graph of $y = \sqrt{r^2 x^2}$ about the *x*-axis. Use the formula for surface area, $S = 4\pi r^2$, to find the centroid of the semicircle $y = \sqrt{r^2 x^2}$.
- 56. A torus is formed by revolving the graph of $(x 1)^2 + y^2 = 1$ about the *y*-axis. Find the surface area of the torus.
- **57. Finding a Centroid** Let $n \ge 1$ be constant, and consider the region bounded by $f(x) = x^n$, the *x*-axis, and x = 1. Find the centroid of this region. As $n \to \infty$, what does the region look like, and where is its centroid?

PUTNAM EXAM CHALLENGE

58. Let V be the region in the cartesian plane consisting of all points (x, y) satisfying the simultaneous conditions |x| ≤ y ≤ |x| + 3 and y ≤ 4. Find the centroid (x̄, ȳ) of V. This problem was composed by the Committee on the Putnam Prize Competition.
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